

ESAME 6.6.2022 PARTE 2

ESERCIZIO 1

$$f(x,y) = \begin{cases} \frac{\lg^3(1+|y+1|) + e^x}{(y+1)^2 + x^2} & \text{se } (x,y) \neq (0,-1) \\ 1 & \text{se } (x,y) = (0,-1) \end{cases}$$

$$D = \mathbb{R}^2$$

La funzione è continua $\forall (x,y) \neq (0,-1)$

per vedere se è continua in $(x,y) = (0,-1)$

devo dimostrare che

$$\lim_{(x,y) \rightarrow (0,-1)} f(x,y) = f(0,-1) = 1 \Leftrightarrow$$

$$\lim_{(x,y) \rightarrow (0,-1)} |f(x,y) - 1| = 0$$

$$\left| \frac{\lg^3(1+|y+1|) + e^x}{(y+1)^2 + x^2} - 1 \right| \leq$$

$$\leq \left| \frac{\lg^3(1+|y+1|)}{(y+1)^2 + x^2} \right| + |e^x - 1| \Rightarrow$$

$$|e^x - 1| \rightarrow 0$$

$$(x, y) \rightarrow (0, -1)$$

$$\left| \frac{\lg^3(1+|y+1|)}{(y+1)^2 + x^2} \right| \leq \frac{|\lg^3(1+|y+1|)|}{(y+1)^2}$$

$$\underset{y \rightarrow -1}{\sim} \frac{|y+1|^3}{(y+1)^2} = |y+1| \rightarrow 0$$

$\Rightarrow f(x, y)$ continua in D .

$$\frac{\partial f}{\partial x}(0, -1) = \frac{d}{dx} f(x, -1) \Big|_{x=0} = \frac{d}{dx} e^x \Big|_{x=0} = 1$$

$$\frac{\partial f}{\partial y}(0, -1) = \frac{d}{dy} f(0, y) \Big|_{y=-1} = \frac{d}{dy} \left(\frac{\lg^3(1+|y+1|)}{|y+1|^2} \right) \Big|_{y=-1}$$

$$= \frac{3 \lg^2(1+|y+1|) \frac{1}{1+|y+1|} \operatorname{sg}(y+1)(y+1)^2 - 2(y+1) \lg^3(1+|y+1|)}{(y+1)^4} \Big|_{y=-1}$$

$$\underset{y \rightarrow -1}{\sim} \frac{3}{1+|y+1|} \operatorname{sg}(y+1) - \frac{2(y+1)|y+1|^3}{(y+1)^4} = \frac{3}{1+|y+1|} \operatorname{sg}(y+1) - 2 \operatorname{sg}(y+1)$$

$$\rightarrow \underset{y \rightarrow -1^\pm}{\begin{cases} 3-2 = 1 \\ -3+2 = -1 \end{cases}}$$

$\Rightarrow$ la derivata non esiste

$$\text{Derivata direzionale } D_{\hat{r}} f(0, -1) =$$

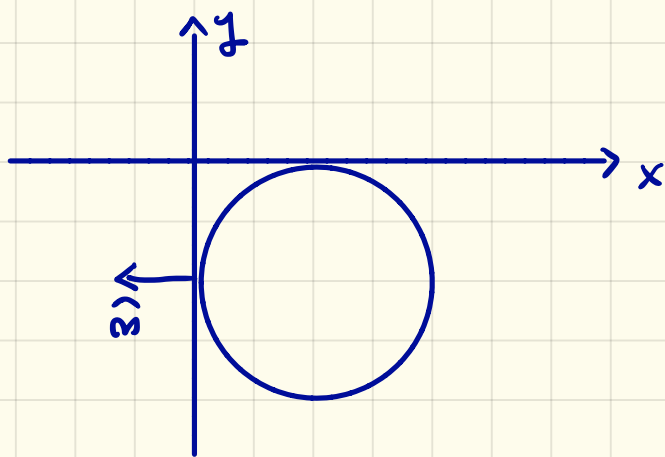
$$= \lim_{t \rightarrow 0} \frac{f(t \cos \theta, -1 + t \sin \theta) - f(0, -1)}{t} =$$

$$= \lim_{t \rightarrow 0} \left[\frac{\lg^3(1 + |t \sin \theta|) + e^{\frac{t \cos \theta}{t}} - 1}{t} \right]$$

$$\sim \lim_{t \rightarrow 0} \frac{|t^3 \sin \theta|}{t^3} + \frac{t \cos \theta}{t} = \sin \theta + \cos \theta$$

$\Rightarrow$ la derivata direzionale è definita solo

per $\sin \theta = 0 \Leftrightarrow \theta = 0, \pi$.



$$\hat{m} = (-1, 0) \quad \theta = \pi$$

$$D_{\hat{r}} f(-1, 0) = -1$$

ESERCIZIO 2

$$f(x, y) = 4(y-1)^2 - (y-1)^4 - 4(y-1)^2 x^2 \quad D = \mathbb{R}^2$$

$$\frac{\partial f}{\partial x} = -8x(y-1)^2 = 0$$

$$\frac{\partial f}{\partial y} = 8(y-1) - 4(y-1)^3 - 8(y-1)x^2 = 4(y-1)[2 - (y-1)^2 - 2x^2] = 0$$

$y = 1$ retta di punti critici

$$x = 0 \quad (y-1)^2 = 2 \Rightarrow y = 1 \pm \sqrt{2} \quad P_1 = (0, 1+\sqrt{2}) \quad P_2 = (0, 1-\sqrt{2})$$

$$\frac{\partial^2 f}{\partial x^2} = -8(y-1)^2 \quad \frac{\partial^2 f}{\partial y^2} = 8 - 12(y-1)^2 - 8x^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -16x(y-1) \quad H_f(x, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 8 - 8x^2 \end{pmatrix}$$

$\det H_f(x, 1) = 0 \Rightarrow$ non posso dire nulla sulla natura dei punti critici.

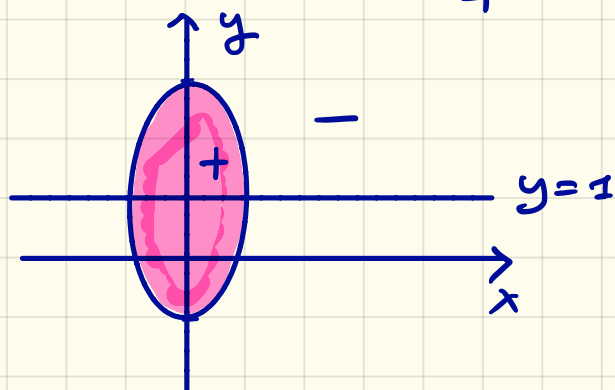
Studio il segno dell'incremento

$$\Delta f(x, y) = f(x, y) - f(x, 1) = f(x, y) \geq 0$$

$$f(x, y) = \underbrace{(y-1)^2}_{\geq 0} (4 - (y-1)^2 - 4x^2) \geq 0 \Rightarrow$$

$$\Leftrightarrow 4 - (y-1)^2 - 4x^2 \geq 0$$

$$1 \geq \frac{(y-1)^2}{4} + x^2 \Rightarrow x^2 + \frac{(y-1)^2}{4} \leq 1$$



$(x, 1)$ punti di max. rel. e locale se $x < -1$
o $x > 1$

$(x, 1)$ punti di sella se $x = -1, x = +1$

$(x, 1)$ punti di min. rel. e locale se $-1 < x < 1$

$$H_f(0, 1 \pm \sqrt{2}) = \begin{pmatrix} -16 & 0 \\ 0 & -16 \end{pmatrix} \Rightarrow P_1 \text{ e } P_2 \text{ sono punti di max locale forte}$$

Matrice diagonale con autovalori negativi

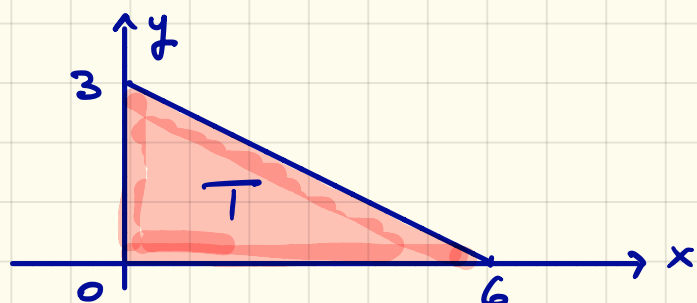
P_1 e P_2 sono anche pti di massimo globale, infatti

$$f(0, 1 \pm \sqrt{2}) = 4 \geq f(x, y) \quad \forall (x, y) \in \mathbb{R}^2$$

$$4 - f(x, y) = 4 - 4(y-1)^2 + (y-1)^4 + 4(y-1)^2 x^2 = (2 - (y-1)^2)^2 + 4(y-1)^2 x^2 \geq 0 \quad \forall (x, y) \in \mathbb{R}^2$$

ESERCIZIO 3

$$I = \int_T (x^4 y - e^{x+y} y) dx dy$$



$$y = -\frac{1}{2}x + 3$$

$$T = \{ (x, y) \in \mathbb{R}^2 : x \in [0, 6], 0 \leq y \leq -\frac{1}{2}x + 3 \}$$

$$I = \int_0^6 \left(\int_0^{-\frac{1}{2}x+3} (x^4 y - e^{x+y} y) dy \right) dx =$$

$$= \int_0^6 \left(x^4 \frac{1}{2} y^2 \Big|_0^{-\frac{1}{2}x+3} - \int_0^{-\frac{1}{2}x+3} e^x (e^y y - \int_0^y e^y dy) \right) dx$$

$$= \int_0^6 x^4 \left(\frac{1}{2} x - 3 \right)^2 - \int_0^6 \left[e^x \left(e^{-\frac{1}{2}x+3} \left(-\frac{1}{2}x + 3 \right) - \left(e^{-\frac{1}{2}x+3} - 1 \right) \right) \right] dx$$

$$= \int_0^6 x^4 \left(\frac{1}{4} x^2 + 9 - 3x \right) dx - \int_0^6 \left(e^{\frac{1}{2}x+3} \left(-\frac{1}{2}x + 2 \right) + e^x \right) dx$$

$$= \int_0^6 \left(\frac{x^6}{4} + 9x^4 - 3x^5 \right) dx - \left[4e^{\frac{1}{2}x+3} + e^x \right]_0^6 + \int_0^6 e^{\frac{1}{2}x+3} x dx =$$

$$\frac{1}{7.4} 6^7 + \frac{9}{5} 6^5 - \frac{3}{6} 6^6 - [4(e^6 - e^3) + e^{6-1}] +$$

$$+ 2 e^{\frac{1}{2}x+3} \Big|_0^6 - 2 e^{\frac{1}{2}x+2} \Big|_0^6 = \dots$$

ESERCIZIO 4

$$\vec{F}(x,y) = \left(\frac{x-1}{4y^2 + (x-1)^2}, \frac{4y}{(x-1)^2 + 4y^2} \right)$$

$$D = \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (1,0)\}$$

D è aperto, illimitato, connesso, non semplicemente connesso

$$\begin{aligned} \nabla \wedge \vec{F} &= \hat{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = \\ &= \hat{k} \left(- \frac{2(x-1)4y}{((x-1)^2 + 4y^2)^2} + \frac{8y(x-1)}{((x-1)^2 + 4y^2)^2} \right) = 0 \end{aligned}$$

$\Rightarrow$ il campo è irrotazionale $\Rightarrow \vec{F}$ è localmente conservativo.

Una condizione necessaria affinché il campo sia conservativo è che

$$\oint_{\gamma} \vec{F} \cdot d\vec{r} = 0$$

$\forall \gamma$ chiusa, semplice regolare e tutta contenuta in D .

Sia $\gamma \subset D' \subset D$ D' semplicemente connesso

$\Rightarrow$ dato che il campo $\vec{F}$ è localmente conservativo
$$\oint_{\gamma} \vec{F} \cdot d\vec{r} = 0$$

Le curve γ scelte, semplici e regolari che non sono contenute in un sottoinsieme semplicemente connesso di D , hanno al loro interno il punto $(1,0)$.

Sia γ una di queste curve, dato che il campo è irrotazionale posso usare i teoremi equivalenti e scrivere

$$\oint_{\gamma} \vec{F} \cdot d\vec{r} = \oint_{\mathcal{C}} \vec{F} \cdot d\vec{r}$$

$\mathcal{C} \Rightarrow$ ellisse $(x-1)^2 + 4y^2 = 1$

$$\mathcal{C} \quad \begin{cases} x = \cos \theta + 1 \\ y = \frac{1}{2} \sin \theta \end{cases} \quad \theta \in [0, 2\pi]$$

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (\cos \theta, 2 \sin \theta) \cdot (-\sin \theta, \frac{1}{2} \cos \theta) d\theta$$

$$= \int_0^{2\pi} (-\cos \theta \sin \theta + 2 \sin \theta \frac{1}{2} \cos \theta) d\theta = 0$$

$$\Rightarrow \oint_{\gamma} \vec{F} \cdot d\vec{r} = 0 \quad \text{pour toute courbe} \\ \text{choisie, simple}$$

supposée contenue
in D.

$\Rightarrow \vec{F}$ est conservatif.

Calculons le potentiel

$$\frac{\partial U}{\partial x} = F_1 = \frac{x-1}{4y^2 + (x-1)^2}$$

$$U(x, y) = \int \frac{x-1}{4y^2 + (x-1)^2} dx = \frac{1}{2} \lg[4y^2 + (x-1)^2] + c(y)$$

$$\frac{\partial U}{\partial y} = \frac{4y}{4y^2 + (x-1)^2} + c'(y) = \frac{4y}{4y^2 + (x-1)^2}$$

$$\Rightarrow c'(y) = 0 \Rightarrow c(y) = C$$

$$U(x, y) = \frac{1}{2} \lg [4y^2 + (x-1)^2]$$

$$\int_A^B \vec{F} \cdot d\vec{r} = U(-1, 1) - U(-2, 0)$$

$$= \frac{1}{2} \lg(4+4) - \frac{1}{2} \lg 9 = \frac{1}{2} \lg \frac{8}{9}$$

=

ESERCIZIO 5

$$\Sigma = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + (y-1)^2 + z^2 \leq 1 \}$$

$$\vec{F} = (x(y-1), (y-1)z, z)$$

$$\partial \Sigma \quad \begin{cases} x = \cos \theta \sin \varphi & \theta \in [0, 2\pi] \\ y = 1 + \sin \theta \sin \varphi & \varphi \in [0, \pi] \\ z = \cos \varphi \end{cases}$$

$$\vec{r}_\varphi = (\cos \theta \cos \varphi, \sin \theta \cos \varphi, -\sin \varphi)$$

$$\vec{r}_\theta = (-\sin \theta \sin \varphi, \cos \theta \sin \varphi, 0)$$

$$\begin{aligned} \vec{r}_\varphi \wedge \vec{r}_\theta &= (\cos \theta \sin^2 \varphi, + \sin \theta \sin^2 \varphi, \cos \varphi \sin \varphi) \\ &= \sin \varphi (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \end{aligned}$$

$$\vec{r}(\varphi, \theta) \in C^1([0, 2\pi] \times [0, \pi])$$

$$\vec{r}_\varphi \wedge \vec{r}_\theta \neq 0 \Leftrightarrow \sin \varphi \neq 0$$

$\Rightarrow$ la superficie ha due punti di non

superficie $\varphi=0, \varphi=\pi$.

$$\hat{m}_e = \frac{\vec{r}_\varphi \wedge \vec{r}_\theta}{|\vec{r}_\varphi \wedge \vec{r}_\theta|}$$

$$\iint_{\partial \Sigma} \vec{F} \cdot \hat{n}_e \, dS = \int_0^\pi \int_0^{2\pi} \vec{F}(\vec{r}(\varphi, \theta)) \cdot \vec{r}_\varphi \wedge \vec{r}_\theta \, d\varphi d\theta$$

$$= \int_0^\pi \int_0^{2\pi} (\cos\theta \sin^2\varphi \sin\theta, \sin\theta \sin\varphi \cos\theta, \cos\varphi) \cdot \vec{r}_\varphi \wedge \vec{r}_\theta \, d\varphi d\theta$$

$$= \int_0^\pi \int_0^{2\pi} (\cos^2\theta \sin\theta \sin^4\varphi + \sin^2\theta \sin\varphi^3 \cos\varphi + \cos^3\varphi \sin\varphi) \, d\varphi d\theta$$

$$= -\frac{1}{3} \cos^3\theta \Big|_0^{2\pi} \int_0^\pi (1 - \cos 2\varphi)^2 \, d\varphi +$$

$$+ \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} \, d\theta \frac{1}{4} \sin^4\varphi \Big|_0^\pi - 2\pi \frac{1}{3} \cos^3\varphi \Big|_0^\pi$$

$$= -\frac{2\pi}{3} (-1 - 1) = \frac{4}{3}\pi$$

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \vec{F} = (y^{-1} + z + 1)$$

$$\iiint (4+z) \, dx \, dy \, dz =$$

$$= \sum \int_0^1 \int_0^{2\pi} \int_0^{\pi} (1 + \rho \sin \theta \sin \varphi + \rho \cos \varphi) \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi$$

$$\begin{cases} x = \rho \cos \theta \sin \varphi \\ y = 1 + \rho \sin \theta \sin \varphi \\ z = \rho \cos \varphi \end{cases} \quad \begin{cases} \rho \in [0, 1] \\ \theta \in [0, 2\pi] \\ \varphi \in [0, \pi] \end{cases}$$

$$= \frac{1}{3} \rho^3 \Big|_0^1 \int_0^{2\pi} \int_0^{\pi} (\cos \varphi) \, d\varphi \, d\theta + \frac{1}{4} \rho^3 \Big|_0^1 \int_0^{2\pi} \int_0^{\pi} (-\cos \varphi) \, d\varphi \, d\theta$$

$$+ \frac{1}{4} \rho^4 \frac{1}{2} \sin \varphi \Big|_0^{\pi} = \frac{4}{3} \pi$$

ESERCIZIO 6

$$\vec{F}(x, y, z) = (z, x-1, y)$$

$$S = \{ (x, y, z) \in \mathbb{R}^3 : z = 4 - (x-1)^2 - y^2 \quad 2 \leq z \leq 4 \}$$

$$\begin{cases} x = x \\ y = y \\ z = 4 - (x-1)^2 - y^2 \end{cases} \quad 0 \leq (x-1)^2 + y^2 \leq 2$$

$$\nabla \wedge \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} =$$

$$= \hat{i}(1-0) - \hat{j}(0-1) + \hat{k}(1-0) = \hat{i} + \hat{j} + \hat{k}$$

$$\begin{cases} x = 1 + \rho \cos \theta \\ y = \rho \sin \theta \\ z = 4 - \rho^2 \end{cases} \quad \rho \in [0, \sqrt{2}] \quad \theta \in [0, 2\pi]$$

$$\vec{r}_\rho = (\cos \theta, \sin \theta, -2\rho)$$

$$\vec{r}_\theta = (-\rho \sin \theta, \rho \cos \theta, 0)$$

$$\vec{r}_\rho \wedge \vec{r}_\theta = (2\rho^2 \cos \theta, 2\rho^2 \sin \theta, \rho)$$

$$\iint_S \nabla \wedge \vec{F} \cdot \hat{n} \, dS = \int_0^{\sqrt{2}} \int_0^{2\pi} (1, 1, 1) \cdot (2\rho^2 \cos \theta, 2\rho^2 \sin \theta, \rho) \, d\rho \, d\theta$$

$$= \int_0^{\sqrt{2}} \int_0^{2\pi} (2\rho^2 (\cos\theta + \sin\theta) + \rho) d\rho d\theta$$

$$= \frac{1}{2} \rho^2 \Big|_0^{\sqrt{2}} \cdot 2\pi = 2\pi$$

TEOREMA di STOKES

$$\iint_S \nabla \wedge \vec{F} \cdot \hat{n} dS = \int_{\partial S^+} \vec{F} \cdot d\vec{r}$$

$\partial S^+ \Rightarrow (x-1)^2 + y^2 = 2$ percorso in senso
antiorario

$$\begin{cases} z = 2 \\ x = 1 + \sqrt{2} \cos\theta \\ y = \sqrt{2} \sin\theta \end{cases} \quad \theta \in [0, 2\pi]$$

$$\int_{\partial S^+} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (2, \sqrt{2} \cos\theta, \sin\theta) \cdot (-\sqrt{2} \sin\theta, \sqrt{2} \cos\theta, 0) d\theta$$

$$= \int_0^{2\pi} (-2\sqrt{2} \sin\theta + 2 \cos^2\theta) d\theta = 2 \int_0^{2\pi} 1 + \frac{\cos 2\theta}{2} d\theta$$

$$= 2\pi$$